MATH 147: SPRING 2021 EXAM 2 SOLUTIONS

Calculation Problems.

1. Using a Lagrange multiplier, find the point P on the plane $2x+4y+6z=32$ such that the square of the distance from P to the point $(1,-1,1)$ is the least among all points on the plane and determine the corresponding value. What does this tell you about the least distance from points on the plane to (1, -1, 1)? (20 points)

Solution. We need to minimize the function $f(x,y,z)=(x-1)^2+(y+1)^2+(z-1)^2$ subject to the constraint $g(x,y,z)=2x+4y+6z=32$. Setting $\nabla f=\lambda\nabla g$, we have

$$\begin{aligned}2(x-1)&=\lambda 2\\ 2(y+1)&=\lambda 4\\ 2(z-1)&=\lambda 6.\end{aligned}$$

from which we see, $x=\lambda+1, y=2\lambda-1, z=3\lambda+1$. Substituting into the constraint equation, we have,

$$2(\lambda+1)+4(2\lambda-1)+6(3\lambda+1)=32.$$

From this, we easily get $\lambda=1$. Thus, $x=2, y=1, z=4$. Therefore, (2, 1, 4) is the point on the plane closest to (1, -1, 1) and the least distance squared we seek is

$$f(2,1,4)=(2-1)^2+(1+1)^2+(4-1)^2=14.$$

Note that what this shows is that the point (2,1,4) is the point on the given plane closest to (1,-1, 1) and the distance between these pints is $\sqrt{14}$.

2. Find the volume of the region in $\mathbb{R}^3$ between the planes $z=x$ and $z=y$ that lies over the square $D:[-1,1]\times[-1,1]$. (20 points).

Solution. We first note that for points (x,y) in the xy-plane, the plane $z=y$ lies above the plane $z=x$, if (x,y) lies above the line $y=x$, and the plane $z=x$ lies above the plane $z=y$, if (x,y) lies below the line $y=x$. Thus, the volume we seek is

$$\int_{-1}^{1}\int_{x}^{1} y-x\ dy\ dx+\int_{-1}^{1}\int_{-1}^{x} x-y\ dy\ dx.$$

Working the first of the double integrals we get

$$\begin{aligned}\int_{-1}^{1}\int_{x}^{1} y-x\ dy\ dx&=\int_{-1}^{1}(\frac{1}{2}y^2-xy)\Big|_{y=x}^{y=1}\ dx\\ &=\int_{-1}^{1}(\frac{1}{2}-x)-(\frac{1}{2}x^2-x^2)\ dx\\ &=\int_{-1}^{1}\frac{1}{2}-x+\frac{x^2}{2}\ dx\\ &=(\frac{x}{2}-\frac{x^2}{2}+\frac{x^3}{6})\Big|_{-1}^{1}\\ &=\frac{4}{3}.\end{aligned}$$

A similar calculation (or symmetry) shows that the second double integral above is also $\frac{4}{3}$. Thus, the volume of the specified region is $\frac{8}{3}$.

3. Calculate $\int\int_D (x^2+y^2)^{-2}\ dA$, where $D : \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 9\}$. (15 points)

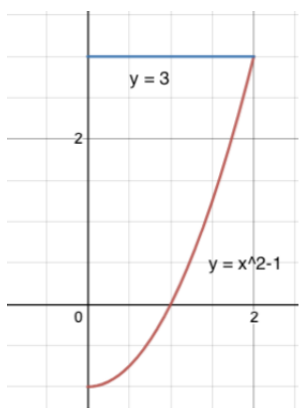
Solution. Let D_R denote the region in $\mathbb{R}^2$ define by: $9 \leq x^2+y^2 \leq R^2$. Then, if the limit below exists, it equals $\int\int_D (x^2+y^2)^{-2}\ dA$.

$$\begin{aligned}
\lim_{R\to\infty} \int\int_{D_R} (x^2+y^2)^{-2}\ dA &= \lim_{R\to\infty} \int_0^{2\pi} \int_3^R ((r\sin(\theta))^2 + (r\sin(\theta))^2)^{-2}\ r\ dr\ d\theta \\
&= \lim_{R\to\infty} \int_0^{2\pi} \int_3^R r^{-3}\ dr\ d\theta \\
&= \lim_{R\to\infty} \int_0^{2\pi} -\frac{r^{-2}}{2}\Big|_{r=3}^{r=R}\ d\theta \\
&= \lim_{R\to\infty} 2\pi \cdot (-\frac{R^{-2}}{2} + \frac{1}{18}) \\
&= \frac{\pi}{9}.
\end{aligned}$$

Short Answer.

4. Explain how you would calculate $\int_0^2 \int_{x^2-1}^3 e^{(y+1)^{\frac{3}{2}}}\ dy\ dx$ and why this works, though you do not have to calculate a double integral. (15 points)

Solution. As given, we cannot antidifferentiate the integrand with respect to y. Since the given integral is $\int\int_D e^{(y+1)^{\frac{3}{2}}}\ dA$, for D



by Fubini's theorem we can change the order of integration, thereby getting $\int_0^3 \int_0^{\sqrt{y+1}} e^{(y+1)^{\frac{3}{2}}}\ dx\ dy$. When we integrate with respect to x we get

$$\int_0^2 \{xe^{(y+1)^{\frac{3}{2}}}\}\Big|_{x=0}^{x=\sqrt{y+1}}\ dy = \int_0^3 \sqrt{y+1}e^{(y+1)^{\frac{3}{2}}}\ dy,$$

which can easily be solved by u-substitution.

5. Let $G(u,v) = (3u+5v+1, 7u+2v+4)$ be a transformation from the uv-plane to the xy-plane.

(i) If D_0 is the square $[-1,1]\times[-1,1]$ in the uv-plane, describe the region D in the xy-plane obtained by applying $G(u,v)$ to D_0. You do not have to justify your answer. (7.5 points)

(ii) If $F(x,y)$ is the inverse transformation of $G(u,v)$, find Jac(F).(7.5 points)

Solution. For Part (i), The transformation $G(u,v)$ is the linear transformation $T(u,v) = (3u+5v, 7u+2v)$ followed by a translation that takes the origin to (1,4). Thus, D is the parallelogram (centered at (1,4)) with vertices (9, 13), (-1,9), (3, -1), (-7, -5), which are obtained by evaluating $G(u,v)$ at the vertices of D_0.

For (ii), $\text{Jac}(G) = \det \begin{pmatrix} 3 & 5 \\ 7 & 2 \end{pmatrix} = -29$, therefore $\text{Jac}(F) = \frac{1}{\text{Jac}(G)} = -\frac{1}{29}$.

6. For the region D bounded by the lines $y = 0, y = \frac{x}{2}$ and $x + y = 1$, find a change of variables that makes it possible to calculate $\int\int_D \sqrt{\frac{x+y}{x-2y}}\ dA$, and then set up the resulting double integral. Do not calculate the double integral. (15 points)

Solution. We first note that D is the triangle in the xy-plane with vertices (0,0), (1, 0), $(\frac{2}{3}, \frac{1}{3})$. We want to have $u = x + y$ and $v = x - 2y$, which will enable us to antidifferentiate the integrand $\sqrt{\frac{u}{v}}$, so we think of these equations as as the coordinates of $F(x, y)$, the inverse of our change of variables transformation. Solving for x, y in terms of u, v gives $x = \frac{2u+v}{3}$ and $y = \frac{u-v}{3}$. Thus, we take $G(u, v) = (\frac{2u+v}{3}, \frac{u-v}{3})$. Since $x + y = 1$ is one edge of D and $u = x + y$, F transforms this line to $u = 1$. Similarly, since $v = x - 2y$, F transforms the line $y = \frac{x}{2}$ to the line $v = 0$. Finally, when $y = 0$, $u = x = v$, so that F transforms the line $y = 0$ to the line $v = u$. It follows that F transforms D to the triangle in the uv-plane having vertices (0,0), (1,0), (1,1). Since the absolute value of the Jacobian of G is easily seen to be $\frac{1}{3}$, it follows that

$$\int\int_D \sqrt{\frac{x+y}{x-2y}}\ dA = \int_0^1 \int_0^u \sqrt{\frac{u}{v}}\ \frac{1}{3}\ dv\ du.$$

Note that the inner integral is an improper integral, but a convergent proper integral, so it is easy to obtain a final answer.

Optional Bonus Problem. Let D denote the region in $\mathbb{R}^2$ between the ellipses $E_1 : \frac{(x-4)^2}{9} + \frac{(y+7)^2}{16} = 1$ and $E_2 : \frac{(x-4)^2}{36} + \frac{(y+7)^2}{64} = 1$. Calculate $\int\int_D \sqrt{16(x-4)^2 + 9(y+7)^2}\ dA$. (15 points)

Solution. We will transform the ellipses to circles centered at the origin in the uv-plane, and then use polar coordinates. First use the transformation $G(u, v) = (3u + 4, 4v - 7) = (x, y)$, which has $\text{Jac}(G) = 12$. Notice that if we substitute these equations into the equations for the ellipses, E_1 becomes the circle $C_1 : u^2 + v^2 = 1$ and E_2 becomes the circle $C_2 : u^2 + v^2 = 4$. In addition, the integrand becomes $12(u^2 + v^2)^{\frac{1}{2}}$. Thus, if we let D_0 denote the region in the uv-plane between the circles C_1 and C_2, we have

$$\begin{aligned}
\int\int_D \sqrt{16(x-4)^2 + 9(y+7)^2}\ dA &= \int\int_{D_0} 12(u^2 + v^2)^{\frac{1}{2}}\ 12\ dA \\
&= 144 \int_0^{2\pi} \int_1^2 r \cdot r\ dr\ d\theta \\
&= 144 \int_0^{2\pi} \frac{r^3}{3}\Big|_{r=1}^{r=2}\ d\theta \\
&= 144 \cdot \frac{7}{3} \cdot 2\pi \\
&= 672\pi.
\end{aligned}$$